SOLUTION OF A TWO-DIMENSIONAL EQUATION OF NONSTATIONARY HEAT CONDUCTION FOR ORTHOTROPIC CYLINDRICALLY BOUNDED MEDIA

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Analytical relationships are presented to determine two-dimensional (2-D) nonstationary thermal fields in a bounded orthotropic cylinder for various boundary conditions. The solution obtained generalizes a wide class of eighty boundary-value problems of nonstationary heat conduction in studies of heat exchange in the bodies being considered.

The proposed solution of a two-dimensional problem of nonstationary heat conduction for the bounded orthotropic cylinder for different combinations of boundary conditions of the first, second, and third kind is based on the following properties of the infinite, integral Laplace transform:

1. If $L^{-1}[\Phi(s)] = f(\tau)$ and $L^{-1}[\Psi(s)] = \varphi(\tau)$, then $L^{-1}[\Phi(s+A)] = \exp(-A\tau) f(\tau)$;
2. $L^{-1}[\Phi(s)\Psi(s)] = \int\limits_0^\tau f(\xi)\varphi(\tau-\xi)\,d\xi$.

It is known that the solution of the one-dimensional (1-D) problem of nonstationary heat conduction for an unbounded plate with the use of the inverse transform is written in the form [1]

$$\bar{\Theta}^{1\text{-D}}(z,\ s) = \bar{T}(z,\ s) - \frac{T_0}{s} = A\,\mathrm{ch}\left(\frac{z}{\sqrt{a}}\sqrt{s}\right) + B\,\mathrm{sh}\left(\frac{z}{\sqrt{a}}\sqrt{s}\right), \tag{1}$$

and the solution of the two-dimensional nonstationary heat conduction problem for a bounded orthotropic cylinder with the use of the Laplace—Hankel inverse transform is of the form

$$\bar{\Theta}^{2\text{-D}}(p,\ z,\ s) = \bar{T}^{2\text{-D}}(p,\ z,\ s) - \frac{T_0}{s} = A^*\,\mathrm{ch}\left(\frac{z}{\sqrt{a_z}}\sqrt{a_r p^2 + s}\right) + \\ + B^*\,\mathrm{sh}\left(\frac{z}{a_z}\sqrt{a_r p^2 + s}\right) + \varphi(p,\ s), \tag{2}$$

where $\phi(p, s)$ is a function depending on the boundary conditions on the lateral surface of the cylinder; a_z and a_r are the temperature conductivities in the direction of the z- and r-axes; T_0 = const is the initial temperature of the solids under consideration; p and s are the parameters of the finite integral Laplace—Hankel transform [1, 2].

By considering solutions (1) and (2), it is easy to note that the value of the complex $\sqrt{a_r p^2 + s}$ for the two-dimensional case (at $a = a_z$) is different from the value of the complex $\sqrt{s}$ for the one-dimensional case.

Consequently, if the solution is known for the one-dimensional problem of heat conduction in an unbounded plate of thickness 2h for the given boundary conditions on its surfaces, then it is not difficult to obtain a solution of the two-dimensional problem for a bounded cylinder (disk) of diameter 2R for the same boundary conditions on the end planes and for the specific boundary condition on the lateral surface of the cylinder. For boundary conditions of the third kind on the end and lateral surfaces of the cylinder the heat-exchange-coefficients in the general case can be diffeent ($\alpha_h \neq \alpha_R$).

If we represent the combinations of boundary conditions as they are schematically depicted in Fig. 1, then the one-dimensional solutions can be designated as $\Theta_\nu^{2\text{-D}}(z,\ \tau)$ (ν = 1-6) and the two-dimensional solutions, $\Theta_{\nu,\mu}^{2\text{-D}}(r,\ z,\ \tau)$ (μ = 1, 2, 3), with the values of ν and μ being assigned the following meaning:

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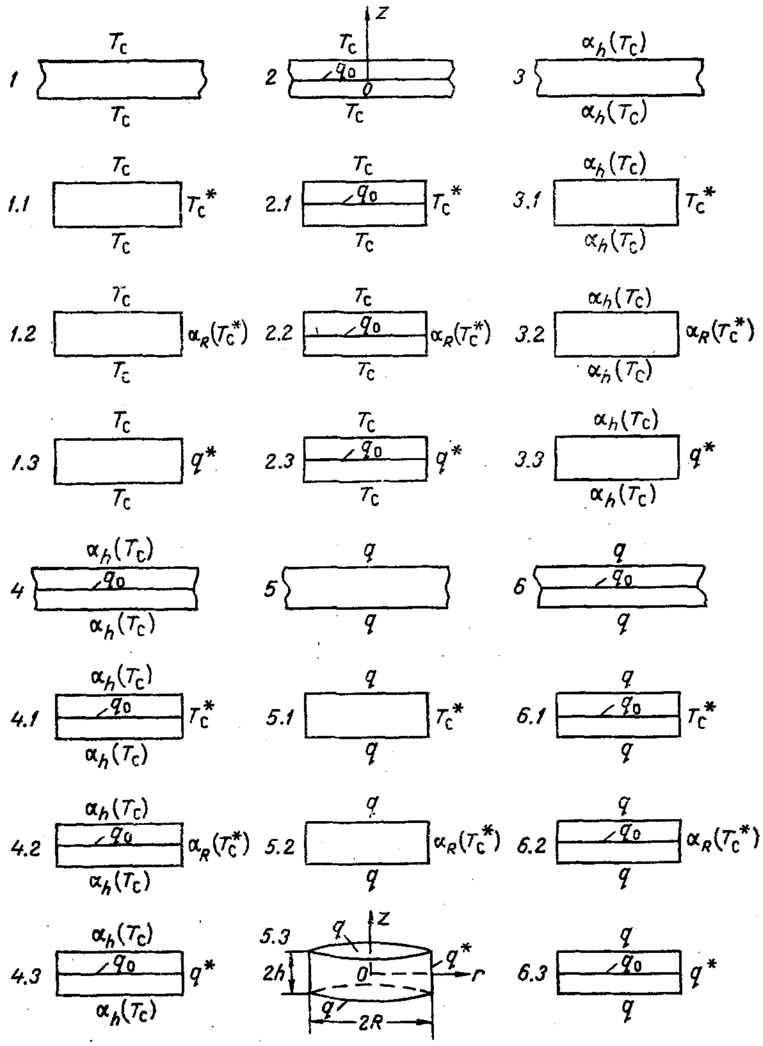

Fig. 1. A schematic representation of combinations of the boundary conditions on the surfaces of an unbounded plate and a bounded orthotropic cylinder.

ν = 1, 2, the source of heat of constant specific intensity q_0 is absent of acts respectively, in the plane z = 0, and the end planes (z = ±h) are maintained at a constant temperature T_c different from the initial temperature T_0;

ν = 3, 4, the source of heat of constant specific intensity q_0 is absent or acts, respectively in the plane z = 0, and the end surfaces (z = ±h) exchange heat according to Newton's law with the ambient medium at the temperature $T_c = T_0$ with the heat-exchange coefficient α_h;

ν = 5, 6, the source of heat of the constant specific capacity q_0 is absent or acts, respectively, in the plane z = 0, and the end surfaces (z = ±h) are heated by the constant temperature flow of density $q \neq q_0$;

μ = 1, 2, 3, respectively, on the lateral surface (r = R) of the orthotropic cylinder the constant temperature $T_c^* \neq T_c \neq T_0$ is maintained; heat is exchanged according to Newton's law with a medium having the temperature T_c^* with the heat-exchange coefficient $\alpha_R \neq \alpha_h$; a source of heat of constant specific capacity $q^* \neq q \neq q_0$ acts.

We assume that the origin of the cylindrical coordinates is located in the center of the cylinder, and that for the unbounded plate the plane z = 0 is in the middle of the plate. The initial temperature distribution of the bodies under consideration both in the one-dimensional case and in the two-dimensional case is uniform: T_0 = const. The considered solids are assumed to be orthotropic.

From the concept of anisotropy it follows from the body is assumed to be orthotropic if the spatial variation in its thermophysical properties is strictly bounded in the direction

TABLE 1. Values of Γ_μ and H_μ and Characteristic Equations for Calculating the Roots δ_k

Values of	$\mu=1$		$\mu=2$		$\mu=3$	
	Γ_μ	H_μ	Γ_μ	H_μ	Γ_μ	H_μ
$\nu=1$	$\delta_k^2 \frac{a_r (T_c^* - T_0)}{R^2 (T_c - T_0)}$	$T_c^* - T_0$	$\delta_k^2 \frac{a_r (T_c^* - T_0)}{R^2 (T_c - T_0)}$	$T_c^* - T_0$	$\frac{a_r q^* \delta_k J_0(\delta_k)}{\lambda_r R (T_c - T_0) J_1(\delta_k)}$	$\frac{q^* R J_0(\delta_k)}{\lambda_r \delta_k J_1(\delta_k)}$
$\nu=2$	$\delta_k^2 \frac{a_r (T_c^* - T_0)}{R^2 (T_c - T_0)}$	$T_c^* - T_0$	$\delta_k^2 \frac{a_r (T_c^* - T_0)}{R^2 (T_c - T_0)}$	$T_c^* - T_0$	$\frac{a_r q^* \delta_k J_0(\delta_k)}{\lambda_r R (T_c - T_0) J_1(\delta_k)}$	$\frac{q^* R J_0(\delta_k)}{\lambda_r \delta_k J_1(\delta_k)}$
$\nu=3$	$\delta_k^2 \frac{a_r (T_c^* - T_0)}{R^2 (T_c - T_0)}$	$T_c^* - T_0$	$\delta_k^2 \frac{a_r (T_c^* - T_0)}{R^2 (T_c - T_0)}$	$T_c^* - T_0$	$\frac{a_r q^* \delta_k J_0(\delta_k)}{\lambda_r R (T_c - T_0) J_1(\delta_k)}$	$\frac{q^* R J_0(\delta_k)}{\lambda_r \delta_k J_1(\delta_k)}$
$\nu=4$	$\delta_k^2 \frac{a_r (T_c^* - T_0)}{R^2 (T_c - T_0)}$	$T_c^* - T_0$	$\delta_k^2 \frac{a_r (T_c^* - T_0)}{R^2 (T_c - T_0)}$	$T_c^* - T_0$	$\frac{a_r q^* \delta_k J_0(\delta_k)}{\lambda_r R (T_c - T_0) J_1(\delta_k)}$	$\frac{q^* R J_0(\delta_k)}{\lambda_r \delta_k J_1(\delta_k)}$
$\nu=5$	0	$T_c^* - T_0$	0	$T_c^* - T_0$	0	$\frac{q^* R J_0(\delta_k)}{\lambda_r \delta_k J_1(\delta_k)}$
$\nu=6$	0	$T_c^* - T_0$	0	$T_c^* - T_0$	0	$\frac{q^* R J_0(\delta_k)}{\lambda_r \delta_k J_1(\delta_k)}$
Characteristic equation for	$J_0(\delta_k) = 0$ for all ν and $\mu = 1$		$\frac{J_0(\delta_k)}{J_1(\delta_k)} = \frac{\delta_k}{\mathrm{Bi}_R}$ for all ν and $\mu = 2$		$J_1(\delta_k) = 0$ for all ν and $\mu = 3$	

coinciding with the unit vectors of the selected orthogonal system of coordinates. In this case we consider cylindrical anisotropy, i.e., only in the direction of the r- and z-coordinates do components of heat conduction (λ_r and λ_z), temperature conductivity (a_r and a_z), and temperature activity (b_r and b_z) differ one another. Specific heats in the direction of the r- and z-coordinates are assumed to be equal, i.e., $c_r\gamma_z = c_z\gamma_z$. From the last assumption it is not difficult to derive the following relationship between the thermophysical characteristics for an orthotropic cylindrical body:

$$\frac{\lambda_r}{\lambda_z} = \frac{a_r}{a_z}, \tag{3}$$

i.e., the ratio of the heat conduction components for an orthotropic body in the direction of the cylindrical r- and z-coordinates is equal to the corresponding ratio of the temperature conductivities in the same directions.

Thus, taking account of the foregoing, the general solution of a two-dimensional nonstationary equation of heat conduction for an orthotropic bounded cylindrical can be written in the form

$$\Theta_{\nu,\mu}^{2\text{-D}}(r, z, \tau) = \sum_{k=1}^{\infty} A_k \left\{ \exp\left[-\delta_k^2 \frac{a_r\tau}{R^2}\right] \Theta_\nu^{1\text{-D}}(z, \tau) + \delta_k^2 \frac{a_r}{R^2} \int_0^\tau \exp\left[-\delta_k^2 \frac{a_r}{R^2}\xi\right] \Theta_\nu^{1\text{-D}}(z, \xi)\, d\xi - \right.$$

$$\left. - \Gamma_\mu \int_0^\tau \exp\left[-\delta_k^2 \frac{a_r}{R^2}\xi\right] \Theta^{1\text{-D}}_{\nu - \frac{1}{2} - \frac{1}{2}(-1)^\nu}(z, \xi)\, d\xi + H_\mu \left[1 - \exp\left(-\delta_k^2 \frac{a_r\tau}{R^2}\right)\right] \right\}, \tag{4}$$

where

$$A_k = \frac{2J_0\left(\delta_k \frac{r}{R}\right) J_1(\delta_k)}{\delta_k \left[J_0^2(\delta_k) + J_1^2(\delta_k)\right]}. \tag{5}$$

The multipliers Γ_μ, H_μ and the characteristic equations for calculating the roots δ_k, depending on the boundary conditions on the lateral surface of the cylinder, are given in Table I. Solutions of the one-dimensional problems for $\nu = 1$, 3, and 5 are given in [1], while solutions for $\nu = 2$, 4, and 6 have been obtained by the authors:

$$\Theta_1^{1\text{-D}}(z, \tau) = (T_c - T_0)\left[1 - \sum_{n=1}^{\infty} \frac{2\cos\mu_n \frac{z}{h}}{\mu_n \sin\mu_n} \exp\left(-\mu_n^2 \frac{a_z\tau}{h^2}\right)\right] \tag{6}$$

(μ_n are the roots of the equation $\cos\mu_n = 0$);

$$\Theta_2^{1\text{-D}}(z, \tau) = \Theta_1^{1\text{-D}}(z, \tau) + \frac{q_0 h}{\lambda_z}\left[1 - \frac{z}{h} - \sum_{n=1}^{\infty} \frac{2\cos\mu_n \frac{z}{h}}{\mu_n^2} \times \right.$$

$$\left. \times \exp\left(-\mu_n^2 \frac{a_z\tau}{h^2}\right)\right] \tag{7}$$

(μ_n are the roots of the equation $\cos\mu_n = 0$);

$$\Theta_3^{1\text{-D}}(z, \tau) = (T_c - T_0)\left[1 - \sum_{n=1}^{\infty} \frac{2\cos\mu_n \frac{z}{h}\sin\mu_n}{\mu_n + \cos\mu_n \sin\mu_n} \exp\left(-\mu_n^2 \frac{a_z\tau}{h^2}\right)\right] \tag{8}$$

(μ_n are the roots of the equation $\cot\mu_n = \mu_n/Bi_h$);

$$\Theta_4^{1\text{-D}}(z,\ \tau)=\Theta_3^{1\text{-D}}(z,\ \tau)+\frac{q_0 h}{\lambda_z}\left[1+\frac{1}{\mathrm{Bi}_h}-\frac{z}{h}-\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{2\cos\mu_n\dfrac{z}{h}}{\mu_n(\mu_n+\cos\mu_n\sin\mu_n)}\exp\left(-\mu_n^2\frac{a_z\tau}{h^2}\right)\right] \tag{9}$$

(μ_n are the roots of the equation $\cot\mu_n = \mu_n/\mathrm{Bi}_h$);

$$\Theta_5^{1\text{-D}}(z,\ \tau)=\frac{qh}{\lambda_z}\left[\frac{a_z\tau}{h^2}-\frac{h^2-3z^2}{6h^2}+2\sum_{n=1}^{\infty}(-1)^{n+1}\times\frac{\cos\mu_n\dfrac{z}{h}}{\mu_n^2}\exp\left(-\mu_n^2\frac{a_z,\ \tau}{h^2}\right)\right] \tag{10}$$

(μ_n are the roots of the equation $\sin\mu_n = 0$);

$$\Theta_6^{1\text{-D}}(z,\ \tau)=\Theta_5^{1\text{-D}}(z,\ \tau)+\frac{q_0 h}{\lambda_z}\left[\frac{a_z\tau}{h^2}-\frac{h^2-3(h-z)^2}{6h^2}+\right.$$

$$\left.+2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_n-\mu_n\dfrac{z}{h}\right)}{\mu_n^2}\exp\left(-\mu_n^2\frac{a_z\tau}{h^2}\right)\right] \tag{11}$$

(μ_n are the roots of the equation $\sin\mu_n = 0$).

When the thermophysical characteristics on the z-axis are equal to those on the r-axis, i.e., $\lambda_z = \lambda_r$ and $a_z = a_r$, Eq. (4) represents a generalized solution of the two-dimensional nonstationary equation of heat conduction for an isotropic cylinder.

It should be noted that from the obtained generalized solution (4) eighty particular solutions are derived for orthotropic and isotropic cylindrical bounded, semi-bounded, and unbounded (plate, cylinder) bodies.

As an example of a practical application of the obtained solution (4) we find the solution of a two-dimensional problem of nonstationary heat conduction for a bounded orthotropic cylinder for the following boundary conditions.

The initial temperture is T_0. On the end surfaces, the constant temperature $T_c \neq T_0$ is defined, while on the lateral surface, the temperature $T_c^* \neq T_c \neq T_0$. In the plane $z = 0$, there acts a source of heat having constant specific power q_0 = const.

It is required to determine a two-dimensional temperature field $T(r, z, \tau)$. By applying solution (4) to the case $\nu = z$, $\mu = 1$, we obtain

$$\Theta_{2,1}(r,\ z,\ \tau)=\sum_{k=1}^{\infty}A_k\left\{\exp\left(-\delta_k^2\frac{a_r\tau}{R^2}\right)\left\{(T_c-T_0)\left[1-\right.\right.\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{2\cos\mu_n\dfrac{z}{h}}{\mu_n\sin\mu_n}\exp\left(-\mu_n^2\frac{a_z\tau}{h^2}\right)\right]+\frac{q_0 h}{\lambda_z}\left[1-\frac{z}{h}-\right.$$

$$\left.\left.-\sum_{n=1}^{\infty}\frac{2\cos\mu_n\dfrac{z}{h}}{\mu_n^2}\exp\left(-\mu_n^2\frac{a_z\tau}{h^2}\right)\right]\right\}+$$

$$+\delta_k^2\frac{a_r}{R^2}\int_0^{\tau}\exp\left(-\delta_k^2\frac{a_r}{R^2}\xi\right)\left\{(T_c-T_0)\left[1-\sum_{n=1}^{\infty}\frac{2\cos\mu_n\dfrac{z}{h}}{\mu_n\sin\mu_n}\times\right.\right.$$

$$\left.\times\exp\left(-\mu_n^2\frac{a_r}{R^2}\xi\right)\right]+\frac{q_0 h}{\lambda_z}\left[1-\frac{z}{h}-\sum_{n=1}^{\infty}\frac{2\cos\mu_n\dfrac{z}{h}}{\mu_n^2}\times\right.$$

$$\left.\left.\times\exp\left(-\mu_n^2\frac{a_z}{h^2}\xi\right)\right]\right\}d\xi-\delta_k^2\frac{a_r}{R^2}\frac{T_c^*-T_0}{T_c-T_0}\int_0^{\tau}\exp\left(-\delta_k^2\frac{a_r}{R^2}\xi\right)\times$$

$$\times (T_c - T_0)\left[1 - \sum_{n=1}^{\infty} \frac{2\cos\mu_n \dfrac{z}{h}}{\mu_n \sin\mu_n} \exp\left(-\mu_n^2 \frac{a_z}{h^2}\xi\right)\right] d\xi + (T_c^* - T_0)\left[1 - \exp\left(-\delta_k^2 \frac{a_r \tau}{R^2}\right)\right]\Bigg\}, \quad (12)$$

where δ_k are the roots of the equation $J_0(\delta_k) = 0$; μ_n are the roots of the equation $\cos\mu_n = 0$, ie., $\mu_n = (2n - 1)\pi/2$.

After integrating and arranging like terms, Eq. (12) assumes the form

$$\Theta_{2,1}(r, z, \tau) = \sum_{k=1}^{\infty} A_k \Bigg\{(T_c - T_0)\left[1 - 2\sum_{n=1}^{\infty} \frac{\cos\mu_n \dfrac{z}{h}}{\left(1 + \dfrac{\mu_n^2}{\delta_k^2} K_a^{-1} K_R^{-2}\right)\mu_n \sin\mu_n} - \right.$$

$$-2(T_c - T_0)\sum_{n=1}^{\infty} \frac{\mu_n \cos\mu_n \dfrac{z}{h}}{(\delta_k^2 K_a K_R^2 + \mu_n^2)\sin\mu_n} \exp\left[-\left(\delta_k^2 \frac{a_r}{R^2} + \mu_n^2 \frac{a_z}{h^2}\right)\tau\right]\Bigg\} +$$

$$+ \sum_{k=1}^{\infty} A_k \Bigg\{\frac{q_0 h}{\lambda_z}\left[1 - \frac{z}{h} - 2\sum_{n=1}^{\infty} \frac{\cos\mu_n \dfrac{z}{h}}{\left(1 + \dfrac{\mu_n^2}{\delta_k^2} K_a^{-1} K_R^{-2}\right)\mu_n^2}\right] -$$

$$-\frac{2q_0 h}{\lambda_z}\sum_{n=1}^{\infty} \frac{\cos\mu_n \dfrac{z}{h}}{\delta_k^2 K_a K_R^2 + \mu_n^2} \exp\left[-\left(\delta_k^2 \frac{a_r}{R^2} + \mu_n^2 \frac{a_z}{h^2}\right)\tau\right]\Bigg\} +$$

$$+ \sum_{k=1}^{\infty} A_k \Bigg\{2(T_c^* - T_0)\sum_{n=1}^{\infty} \frac{\cos\mu_n \dfrac{z}{h}}{\left(1 + \dfrac{\mu_n^2}{\delta_k^2} K_a^{-1} K_R^{-2}\right)\mu_n \sin\mu_n} \times \left[1 - \exp\left[-\left(\delta_k^2 \frac{a_r}{R^2} + \mu_n^2 \frac{a_z}{h^2}\right)\tau\right]\right]\Bigg\}, \quad (13)$$

where $K_a = a_r/a_z$, $K_R = h/R$, μ_n are the roots of the equation $\cos\mu_n = 0$; δ_k are the roots of the equation $J_0(\delta_k) = 0$, A_k is defined by Eq. (5).

Letting $\lambda_r = \lambda_z = \lambda$, $a_r = a_z = a$ and $T_c^* = T_0$, we obtain an expression for the temperature field of an isotropic bounded cylinder, given in [2], assuming there $Bi_h \to \infty$. For $R \to \infty (T_c^* = T_0)$ from (13) we obtain a one-dimensional solution for an unbounded plate $\Theta_2^{(1-D)}(z, \tau)$.

Thus the obtained two-dimensional solution (4) describes a nonstationary temperature field at any point of a bounded orthotropic cylinder 2h is height and 2R in diameter, if different combinations of boundary conditions of the first, second, and third kind are realized on its boundaries (end surfaces and lateral surfaces); an internal source of heat of constant intensity can act in the central plane of the cylinder (z = 0).

An advantage of the given solution (4) is that for orthotropic cylindrically bounded media the final result in defining a two-dimensional temperature field $T(r, z, \tau)$ is achieved by means of a simple integration of one-dimensional nonstationary solutions for an unbounded plate, and there is no need to solve each time the corresponding problem of nonstationary heat conduction for the bounded orthotropic cylinder (to pass the stage of the definition of the constants of integration for the equation of heat conduction) for the boundary conditions given in Fig. 1.

NOTATION

s, parameter of the indefinite integral Laplace transform; p, parameter of a finite integral Hankel transform; $\Theta_\nu^{1-D}(z, \tau)$ and $\Theta_{\nu,\mu}^{2-D}(r, z, \tau)$, excess temperatures of the one- and two-dimensional nonstationary heat conduction problems for an infinite plate and a bounded cylinder (disk); r, z, coordinates; τ, time; a_r, a_z, λ_r, λ_z, thermal conductivity and heat conduction along the directions of cylindrical coordinates r and z; α_h and α_R, the heat-exchange coefficients on the end and lateral surfaces of the cylinder; $\alpha_h(T_c)$, $\alpha_R(T_c^*)$, the boundary conditions of the third kind (Newton's law) on the end and lateral surfaces of the bounded cylinder at the boundary with the media at temperatures T_c and T_c^*, respectively.

LITERATURE CITED

1. A. V. Lykov, Theory of Heat Conduction [in Russian], Moscow (1967).
2. V. P. Kozlov, Two-Dimensional Axially-Symmetrical Nonstationary Problems of Heat Conduction [in Russian], Minsk (1986).